A NEW CHARACTERIZATION OF FROBENIUS COMPLEMENTS

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ABSTRACT

Let H, G be finite groups such that H acts on G and each non-trivial element of H fixes at most f elements of G. It is shown that, if G is sufficiently large, then H has the structure of a Frobenius complement. This result depends on the classification of finite simple groups. We conclude that, if G is a finite group and $A \subseteq G$ is any non-cyclic abelian subgroup, then the order of G is bounded above in terms of the maximal order of a centralizer $C_G(a)$ for $1 \neq a \in A$.

1. Main results

It is well-known that a finite group H has the structure of a Frobenius complement if and only if it can act fixed-point-freely on some finite group G (this means that non-trivial elements of H fix only the identity element of G). Equivalently, we may require that H acts fixed-point-freely on some elementary abelian group E, or on a linear space V over a field of characteristic zero. The purpose of this note is to derive a characterization of Frobenius complements in terms of *almost* fixed-point-free actions, and to study some of its consequences. For background on Frobenius complements and their basic properties, see [P, Chapter 3].

Let H, G be finite groups and suppose H acts faithfully on G. The **fixity** of this action is defined to be the maximal number of fixed points of a non-trivial element of H. Thus fixed-point-free actions are actions of fixity one. In [Sh] it is shown that, if H acts on G with fixity f, then H has a soluble subgroup of derived length at most 3 whose index is f-bounded (that is, bounded above

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in terms of f alone). It is clear that such a group H need not be a Frobenius complement; in fact it can be shown that H need not even contain a Frobenius complement of f-bounded index.

However, the situation turns out to be quite different if the group G is sufficiently large (relative to H and f). Indeed we have:

THEOREM 1.1: *Let H be a finite group which acts with bounded fixity on finite groups of arbitrarily large order. Then H has the structure of a Frobenius complement.*

Note that the reverse implication is trivial: indeed, if H is a Frobenius complement, let G be a finite group on which H acts fixed-point-freely; then H acts with fixity one on the Cartesian powers G^n for all $n \geq 1$. Hence Theorem 1.1 actually characterizes Frobenius complements.

The proof of this result (implicitly) applies the classification of finite simple groups, and some other tools. It is somewhat intriguing that, though this is a theorem about finite groups, the proof involves some infinite groups at a critical stage. To clarify this point, let H act with fixity f on a very large group G. In order to show that H is a Frobenius complement, it would suffice to find a non-trivial H-invariant section of G on which H acts fixed-point-freely. However, such a section need not exist (for example, consider the case where H, G are p-groups). Therefore, in some situations it seems essential to pass to inverse limits in order to obtain the desired fixed-point-free action.

As a result, the theorem is not effective: while we prove the existence of a function Φ such that, if a group H which is not a Frobenius complement acts on a group G with fixity f then $|G| \leq \Phi(|H|, f)$, we do not obtain any bounds on Φ.

However, in the special case $H = C_p \times C_p$ we do obtain effective bounds, by using an alternative approach. As an application we prove that the order of a finite group is effectively bounded in terms of the fixity of the conjugation action of any non-cyclic abelian subgroup. More specifically we have:

THEOREM 1.2: *Let G be a finite group, and let $A \subseteq G$ be a non-cyclic abelian subgroup. Suppose we have*

$$|C_G(a)| \leq f$$

for all $1 \neq a \in A$. Then

$$|G| \leq \Psi(f),$$

for some function Ψ which may be computed effectively.

In fact we show that we may take

$$\Psi(f) = f^{(3f^{1/3}+1)^2},$$

provided G is nilpotent. See also Proposition 2.1 for more detailed bounds. It is worth mentioning that the bounds obtained here are not best possible: they can all be reduced with some more work.

2. Proofs

Note that if H acts on G with fixity f, then it acts with fixity at most f on any H-invariant section of G. This observation, which follows from [HB, Chapter 8, 10.14(a)], will be applied freely throughout this paper.

Let us prove Theorem 1.1. Suppose H is not a Frobenius complement. Assuming H acts with fixity at most f on a group G, we have to bound the order of G. We start with a series of reductions.

STEP 1: We may assume that G is soluble.

To show this, let p be a prime dividing $|H|$. Then G admits an automorphism of order p with at most f fixed points. By Hartley [H2, 1.2A] (which combines the Classification with a result of Fong [F]) it follows that the soluble radical S of G has p, f-bounded index; in fact, using Fong's arguments, it is possible to obtain an explicit bound on $|G : S|$. In order to bound $|G|$ it therefore suffices to bound $|S|$. Since H acts on S with fixity at most f we may replace G with S.

STEP 2: We may assume that G is nilpotent of class at most two.

It is known that every soluble group G has a nilpotent characteristic subgroup N of class at most two such that $C_G(N) = Z(N)$ (see, for instance, [H1, Lemma 1]). This means that $G/Z(N) \subseteq \mathrm{Aut}(N)$, and so the order of G is bounded above in terms of the order of N. It therefore suffices to bound the order of N.

STEP 3: We may assume that G is abelian.

Since G has class two, $G/Z(G)$ and $Z(G)$ are both abelian, and H acts on them with fixity at most f. It is enough to bound the orders of $G/Z(G)$ and $Z(G)$, so the claim follows.

STEP 4: We may assume that, for some prime $p \le f$, G is an abelian p-group.

Let p be a prime larger than f. If p divides $|G|$ then H acts with fixity less than p on the non-trivial Sylow p-subgroup P of G; hence H acts fixed-point-freely on P and is therefore a Frobenius complement, a contradiction. Therefore the order of G is not divisible by any prime larger than f.

For $p \le f$ let $G(p)$ be the Sylow p-subgroup of G. Since $|G| = \Pi_{p \le f} |G(p)|$ it suffices to bound the orders of each Sylow subgroup $G(p)$. This completes the reduction.

STEP 5: The number of generators $d(G)$ of G is bounded.

Using additive notation, consider the elementary abelian quotient $E = G/pG$. We have to show that $|E|$ is bounded. If p divides $|H|$ let $x \in H$ be an element of order p. Since x has at most f fixed points in E and $(x-1)^p = 0$ in $\mathrm{End}(E)$ we have $|E| \le f^p$ and we are done.

So suppose that H is a p'-group, and consider E as an $\mathbb{F}_p H$-module. As such it is completely reducible. Since there are boundedly many isomorphism types of irreducible $\mathbb{F}_p H$-modules, it suffices to bound the multiplicity of each irreducible module M in the module E.

We claim that these multiplicities are bounded by above $\log_p f$. Indeed, given M, H cannot act fixed-point-freely on M, since it is not a Frobenius complement. Hence some non-trivial element $x \in H$ fixes some non-trivial element of M, and so it fixes at least p elements of M. Hence x fixes at least p^k elements of M^k (the direct sum of k copies of M). We conclude that $M^k \subseteq E$ implies $k \le \log_p f$. The result follows.

STEP 6: The order of G is bounded.

Assuming otherwise, there exists an infinite series G_i of d-generated finite abelian p-groups whose order tends to infinity, such that H acts on each G_i with fixity at most f. We shall use an inverse limit argument to obtain a contradiction.

Consider G_i/pG_i as H-modules. Since these modules have bounded order, they split into finitely many isomorphism classes. By passing to a subsequence we may assume that $G_i/pG_i \cong M_1$ for all i. Similarly, we may assume that the modules G_i/p^2G_i are all isomorphic, say $G_i/p^2G_i \cong M_2$. Proceeding in this manner we can construct an infinite sequence of H-modules M_i with the following properties:

(i) M_i has additive exponent p^i.

(ii) $M_i/p^{i-1}M_i \cong M_{i-1}$ $(i \geq 2)$.

(iii) H acts on each M_i with fixity at most f.

Let M be the inverse limit of these modules. As a p-adic module M is finitely generated. By the well-known structure of finitely generated modules over principal ideal domains it follows that M has a characteristic torsion-free $\mathbb{Z}_p$-submodule T of finite index. We claim that H acts fixed-point-freely on T. Indeed, if $1 \neq a \in T$ is fixed by some $1 \neq x \in H$, then, for each i, x acts trivially on the subgroup generated by the image of a in $M/p^iM \cong M_i$. As these subgroups have unbounded order we obtain a contradiction.

Note that $T \cong \mathbb{Z}_p^k$ for some k. Thus H acts fixed-freely on $\mathbb{Z}_p^k$ and hence on $\mathbb{Q}_p^k$ as well. Therefore H is a Frobenius complement. This contradiction completes the proof of the theorem. ∎

Note that the only use of the Classification is in step 1.

It is natural to ask whether effective bounds can be given on the order of G, given H and f. This can be done in all the steps of the proof, except for step 6 which uses a limit argument. The next result shows that, at least in one particular case, effective bounds can nevertheless be given.

PROPOSITION 2.1: *Let $H \cong C_p \times C_p$ be an elementary abelian group of order p^2, and let G be a finite group acted on by H with fixity f. Then*

(i) *$|G| \leq \Phi(p, f)$ for some function Φ which may be computed effectively.*

(ii) *If G is a p-group we may take $\Phi(p, f) = f^{(3p+1)(p \log_p f+1)}$.*

Proof: Let us first assume that G is abelian. Consider G as an H-module and let $\omega \in \mathbb{C}$ be a primitive pth root of unity. Define

$$V = G \otimes_{\mathbb{Z}} \mathbb{Z}[\omega],$$

considered as an H-module (where H centralizes ω). Then it is easy to see that H acts on V with fixity f^p (as V is isomorphic to a direct sum of p copies of G). Let x, y be generators of H. Consider V as an $\langle x \rangle$-module. Then there exists a submodule $pV \subseteq U \subseteq V$ which is a direct sum of eigenspaces for x; namely,

$$U = \bigoplus_{i=0}^{p-1} U_i,$$

where x acts on U_i as multiplication by ω^i; this follows, e.g., from [HB, Chapter 8, 10.3(c)]. Now, since y commutes with x, each U_i is also a $\langle y \rangle$-module. So, arguing as above, we may find submodules $pU_i \subseteq W_i \subseteq U_i$ such that

$$W_i = \bigoplus_{j=0}^{p-1} W_{ij},$$

where y acts on W_{ij} as multiplication by ω^j.

Let $W = \bigoplus_{i,j=0}^{p-1} W_{ij}$.

Note that each W_{ij} is centralized by some non-trivial element h of H (e.g. take $h = x^j y^{-i}$ if $(i,j) \neq (0,0)$ and $h = x$ otherwise). It follows that

$$W = \sum_L C_W(L),$$

where L ranges over all subgroups of H of order p. Since H acts on W with fixity at most f^p we have

$$|C_W(L)| \leq f^p$$

for all L. Therefore

$$|W| \leq \prod_L |C_W(L)| \leq f^{p(p+1)},$$

as H has $p+1$ subgroups of order p.

Since x acts on the elementary abelian p-group V/pV with fixity at most f^p and $(x-1)^p = 0$ in $\mathrm{End}(V/pV)$ we have

$$|V/pV| \leq f^{p^2}.$$

Similarly, $|pV/p^2V| \leq f^{p^2}$. Therefore $|V/p^2V| \leq f^{2p^2}$. But $p^2V \subseteq W$. Hence

$$|V/W| \leq f^{2p^2}.$$

Putting everything together we obtain

$$|V| \leq f^{2p^2} \cdot f^{p^2+p} = f^{3p^2+p},$$

and so

$$|G| = |V|^{1/p} \leq f^{3p+1}.$$

Part (i) of the proposition now follows, using the reductions employed in the proof of Theorem 1.1. So consider part (ii).

Let G be a p-group and let $N \triangleleft G$ be a maximal abelian normal subgroup. It is well-known that G/N acts faithfully on N, and so

$$|G| = |N||G/N| \leq |N||\mathrm{Aut}(N)| \leq |N|^{d+1}$$

where $d = d(N)$. Since N is abelian the above discussion yields

$$|N| \leq f^{3p+1}.$$

It also follows (using additive notation) that $|N/pN| \leq f^p$ and thus $d(N) \leq p \log_p f$. We conclude that

$$|G| \leq f^{(3p+1)(p \log_p f+1)}.$$

The result follows. ∎

Since p-groups which do not have elementary abelian subgroups of rank two are necessarily cyclic or generalized quaternion, we immediately obtain the following.

COROLLARY 2.2: *Let H, G be finite groups and suppose H acts on G with fixity f. Let p be a prime, and let P be a Sylow p-subgroup of H. Then one of the following holds:*

(i) *P is cyclic.*
(ii) *$p = 2$ and P is generalized quaternion.*
(iii) *$|G| \leq \Phi(p, f)$ where Φ is as above.*

In some cases one may require, in part (iii), that $|G|$ is f-bounded. For example, this is the case when G is a p-soluble group whose order is divisible by p. Indeed, such a group G has a non-trivial characteristic p-section Q, and since P acts on Q with fixity at least p we have $p \leq f$.

Now, let G and A be as in Theorem 1.2. Then A contains an elementary abelian subgroup H of order p^2 for some prime p. Clearly, H acts on G with fixity at most f. Applying the above proposition we conclude that $|G| \leq \Phi(p, f)$. But $f \geq |A| \geq p^2$. Therefore $|G| \leq \Psi(f)$ where

$$\Psi(f) = \Phi(\sqrt{f}, f).$$

Theorem 1.2 is proved.

Let us obtain explicit bounds, under the assumption that G is nilpotent. Let $C_p \times C_p \cong H \subseteq A \subseteq G$ be as above. We first assume that G is a p-group. Note that we may assume $A \cap Z(G) = 1$, for otherwise $|G| \leq f$. Now, since $|A| \geq p^2$ we obtain

$$f \geq |C_G(A)| \geq |AZ(G)| \geq p^3.$$

Thus $p \leq f^{1/3}$. Applying part (ii) of Proposition 2.1 we obtain

$$|G| \leq f^{(3f^{1/3}+1)(p \log_p f+1)} \leq \Psi(f),$$

where

$$\Psi(f) = f^{(3f^{1/3}+1)^2}.$$

Finally, suppose only that G is nilpotent. Write $G = G(p) \times G(p')$, where $G(p)$ is the Sylow p-subgroup of G. Denote the order of $G(p')$ by k. Since H centralizes $G(p')$ we have $k \leq f$; furthermore, H acts on $G(p)$ with fixity at most f/k. By the above discussion we have

$$|G| \leq k \cdot \Psi(f/k) \leq \Psi(f),$$

as required.

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